B.Sc. Semester-IV Examination, 2022-23

MATHEMATICS [Honours]

Course ID : 42111 **Course Code : SH/MTH/401/C-8**

Course Title : Riemann Integration and Series of Functions

Time : 2 Hours Full Marks : 40

The figures in the right-hand margin indicate marks.

Candidates are required to give their answers in their own words as far as practicable.

Notations and symbols have their usual meaning.

UNIT-I

1. Answer any **five** from the following questions:

2×5=10

a) Let $f(x) = \sin x$, when x is rational
$= x$, when x is irrational.

Verify f is Darboux integral or not.

b) Determine $\lim_{x\to 2}\left[\frac{\int_2^x e^{\sqrt{1+t^2}}dt}{x-2}\right]$.

c) Examine the convergence of

$\int_0^\pi \frac{dx}{\cos\alpha-\cos x}; 0 \le \alpha \le \pi$.

d) Let $f_n(x) = x^{n-1} - x^n, x \in [0,1]$. Verify the sequence $\{f_n\}$ is uniformly convergent or not.

[Turn Over]

e) Prove that $\int_0^1 \frac{dx}{(1-x^3)^{\frac{1}{3}}} = \frac{2\Pi}{3\sqrt{3}}$.

f) Find the radius of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+1}(x+1)^n.$$

g) The functions $f, g: [a,b] \to \mathbb{R}$ are both continuous on [a, b] and $\int_a^b |f(x)-g(x)|dx = 0$. Prove that $f = g$.

h) Show that the improper integral $\int_0^\infty \frac{\sin x}{\sqrt{x+x^3}}dx$ is absolutely convergent.

UNIT-II

2. Answer any **four** from the following questions:

5×4=20

a) i) A function f is continuous for all $x \ge 0$ and $f(x) \ne 0$ for all $x > 0$. If $\{f(x)\}^2 = 2\int_0^x f(t)dt$, prove that $f(x) = x$ for all $x \ge 0$.

ii) Let $f(x) = x - [x], x \in [0,3]$. Show that f is Riemann Integrable on [0, 3] and evaluate $\int_0^3 f\,dx$. 3+2

b) i) Prove that the integral $\int_0^\infty \left(\frac{1}{1+x} - \frac{1}{e^x}\right)\frac{1}{x}dx$ is convergent.

ii) Examine the convergence of $\int_0^1 \frac{1}{\sqrt{x(1-x)}}dx$.

3+2

c) $\{f_n\}_{n=2}^\infty$ be a sequence of functions defined by

$$f_n(x) = n^2x, 0 \le x \le \frac{1}{n}$$
$$= -n^2x + 2n, \frac{1}{n} < x < \frac{2}{n}$$
$$= 0, \frac{2}{n} \le x \le 1$$

i) Show that $\{f_n\}_{n=2}^\infty$ converges to a function f on [0,1].

ii) Show that the convergence of the sequence is not uniform on [0,1] by proving that $\lim_{n\to\infty} \int_0^1 f_n \neq \int_0^1 f$. 3+2

d) If $f(x) = \{\pi - |x|\}^2$ on $[-\pi, \pi]$, prove that the Fourier series of f is given by $f(x) = \frac{\pi^2}{3} + \sum_{n=1}^\infty \frac{4}{n^2}\cos nx$. Hence deduce that $\sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$ and $\sum_{n=1}^\infty \frac{1}{n^4} = \frac{\pi^4}{90}$. 3+1+1

e) Prove that a power series can be integrated term by term on any closed and bounded interval contained within the interval of convergence.

5

f) Show that $\int_0^\infty \frac{\sin x}{x}dx$ is conditionally convergent.

5

UNIT-III

3. Answer any **one** of the following questions: 10×1=10

a) i) If f be bounded, integrable and periodic with period 2π in $[-\pi, \pi]$, then

$$\int_{-\pi}^{\pi}[f(x)]^2dx = \left[\frac{a_0^2}{2} + \sum_1^\infty (a_n^2 + b_n^2)\right]$$

ii) Find the interval of convergence of the power series $\sum_{n=0}^\infty \frac{x^n}{2^n n}$.

iii) Use first Mean Value Theorem to prove that

$$\frac{\pi}{6} \le \int_0^{1/2} \frac{1}{\sqrt{(1-x^2)(1-k^2x^2)}}dx \le \frac{\pi}{6}\frac{1}{\sqrt{1-\frac{k^2}{4}}},$$

$k^2 < 1$. 4+3+3

b) i) Let $f_n: [a, b] \to \mathbb{R}$ be Riemann integrable on $[a, b]$. If the sequence $\{f_n\}$ converges uniformly to a function f on $[a, b]$, then show that f is Riemann integrable on $[a, b]$ and $\left\{\int_a^b f_n\right\}$ converges to $\int_a^b f$.

Is the condition of uniform convergence of the sequence $\{f_n\}$ necessary? Justify it.

4+2

ii) Obtain the half-range cosine series for the function f where

$$f(x) \begin{array}{ll} = x & for \quad 0 \le x \le 1/2 \\ = (1-x) & for \quad 1/2 < x \le 1. \end{array}$$ 4

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